The Quantum Harmonic Oscillator

Phys 371
Fall 2020
2 November Lecture

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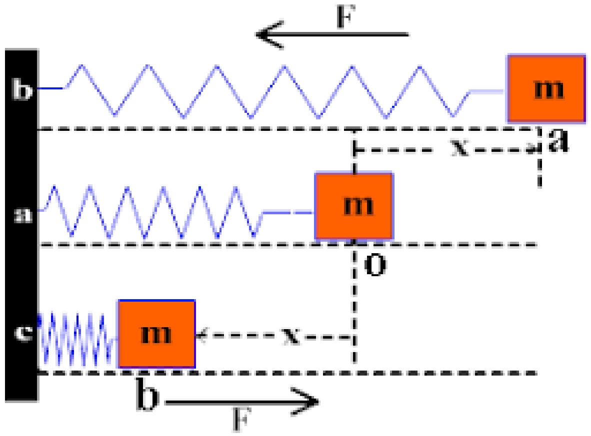
Our objective is to solve the Schrodinger equation for the "mass on a spring" problem, otherwise known as the "harmonic oscillator"

We begin by reviewing some key results about the classical harmonic oscillator

Then we go on to the quantum version and compare to the classical case

The Classical Harmonic Oscillator



No gravity
No friction

The spring is at its equilibrium length when $x = 0$

Pull the spring to the right and it tries to restore its length: For $x > 0, F < 0$ (Left force)

Push the spring to the left and it tries to restore its length: For $x < 0, F > 0$ (Right force)

In equilibrium: For $x = 0, F = 0$ (No force)

This is summarized with Hooke's Law: $F = -kx$

k = spring constant [N/m]

The Classical Harmonic Oscillator

Part 2

If you perturb the spring from equilibrium and let it go, it will execute simple harmonic motion (SHM)

$$\boldsymbol{F = -kx}$$

$$\boldsymbol{F = ma = m\frac{d^2x}{dt^2}}$$

Thus $\quad \boldsymbol{m\frac{d^2x}{dt^2} = -kx} \quad$ **or** $\quad \boldsymbol{\frac{d^2x}{dt^2} = -\frac{k}{m}x \equiv -\omega^2 x}$

$$\boldsymbol{\omega = \sqrt{\frac{k}{m}}}$$

Natural angular frequency of the harmonic oscillator

The solutions to this 2$^{\text{nd}}$-order differential equation are the sines and cosines (or the complex exponentials – whichever is most convenient)

$$\boldsymbol{x(t)} = \boldsymbol{A}\sin(\omega t) + \boldsymbol{B}\cos(\omega t)$$

The Classical Harmonic Oscillator

Part 3

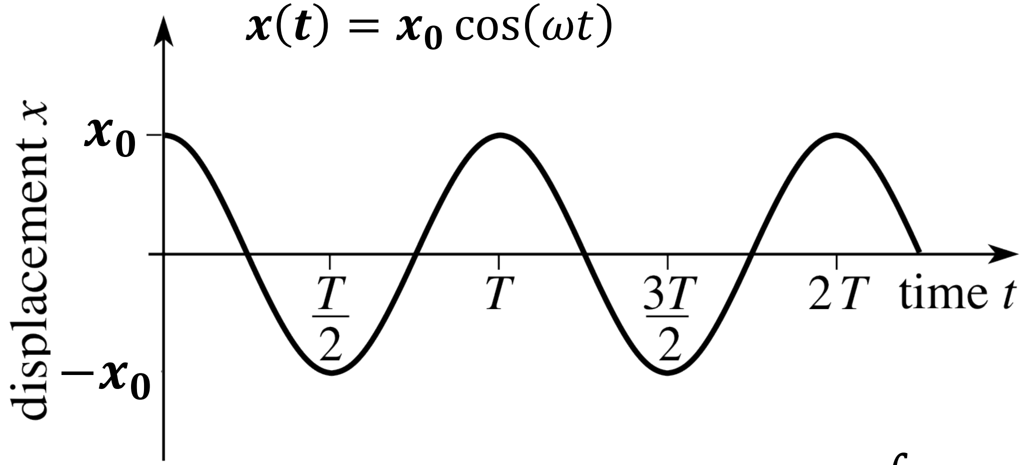
$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{A}\sin(\omega t) + \boldsymbol{B}\cos(\omega t)$$

For example, take as initial conditions: $\boldsymbol{x(0) = x_0}$ **and** $\boldsymbol{\dot{x}(0) = 0}$

This gives B = $\boldsymbol{x_0}$ **and A = 0**

There is no friction or drag at the sub-atomic realm!

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x_0}\cos(\omega t)$$

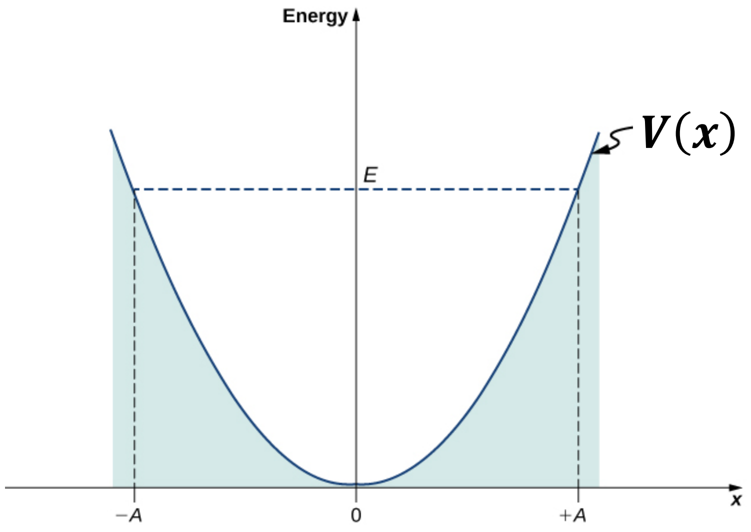


$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

What is the potential energy of the mass on a spring? $\boldsymbol{V(x) = -\int F(x')dx' = +\int_0^x kx'dx' = \frac{1}{2}kx^2}$

What is the kinetic energy of the mass on a spring? $\boldsymbol{T(x) = \frac{1}{2}m\dot{x}^2 \geq 0}$

Quantum Harmonic Oscillator

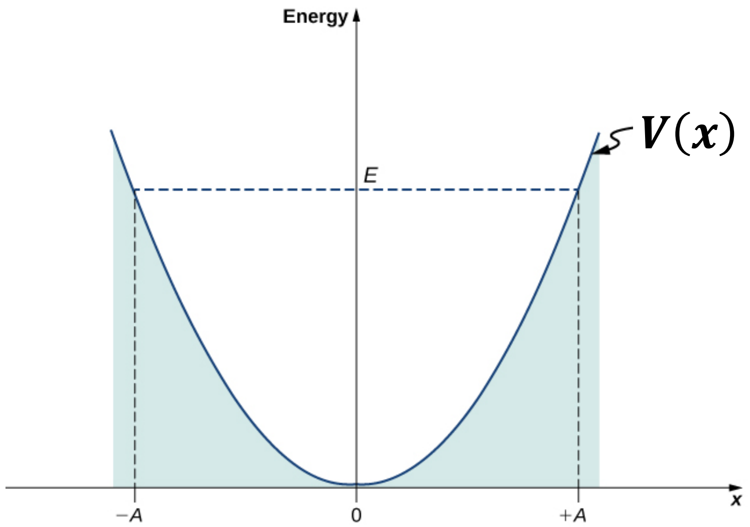


$$V(x) = \frac{1}{2}kx^2$$

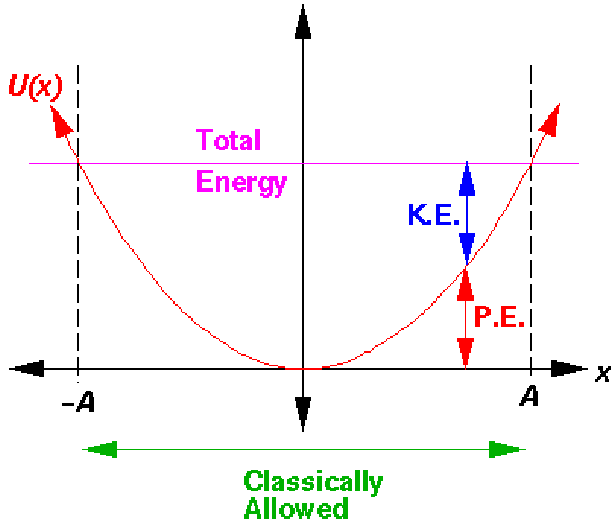
We do NOT solve for the trajectory of a point particle in quantum mechanics

Quantum mechanics is a theory of the wave properties of matter, and waves do not have a well-defined position or location.

Quantum Harmonic Oscillator Potential



$$V(x) = \frac{1}{2}kx^2$$



TISE for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x)$$

using $k = m\omega^2$

This is hard to solve. It will be done in Phys 401. {partial solution: Schrödinger's Trick}

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \text{ where } n = \mathbf{0}, 1, 2, 3, \ldots$$

Note that n starts at 0

$$\psi_n(x) = C_n\, e^{-m\omega x^2/2\hbar}\, H_n(x)$$

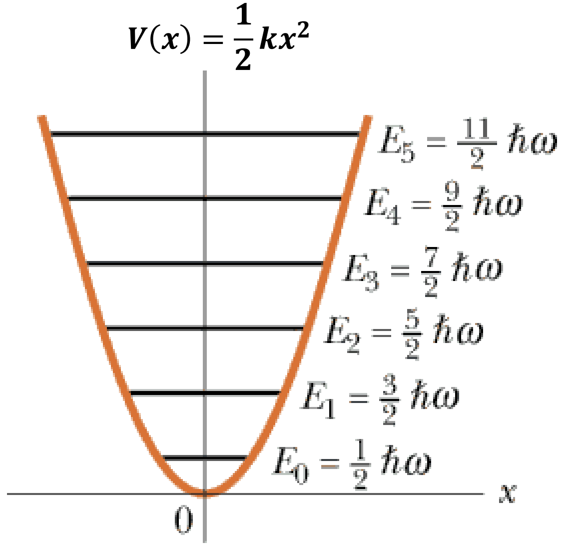
where $H_n(x)$ are the Hermite polynomials of position x

Quantum Harmonic Oscillator Energy Eigenvalues

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where $n = \mathbf{0}, 1, 2, 3, \ldots$

Note that *n* starts at 0

The energy levels are equally spaced



Notice that the minimum energy level is non-zero

$$E_0 = \left(0 + \frac{1}{2}\right)\hbar\omega = \frac{\hbar\omega}{2} > 0$$

The "mass on a spring" does not sit at rest at the bottom of the potential well in its minimum energy state

The ground state energy is non-zero, reflecting "zero-point energy"

Quantum Harmonic Oscillator Eigenfunctions

$$\psi_n(x) = C_n\, e^{-m\omega x^2/2\hbar}\, H_n(x)$$

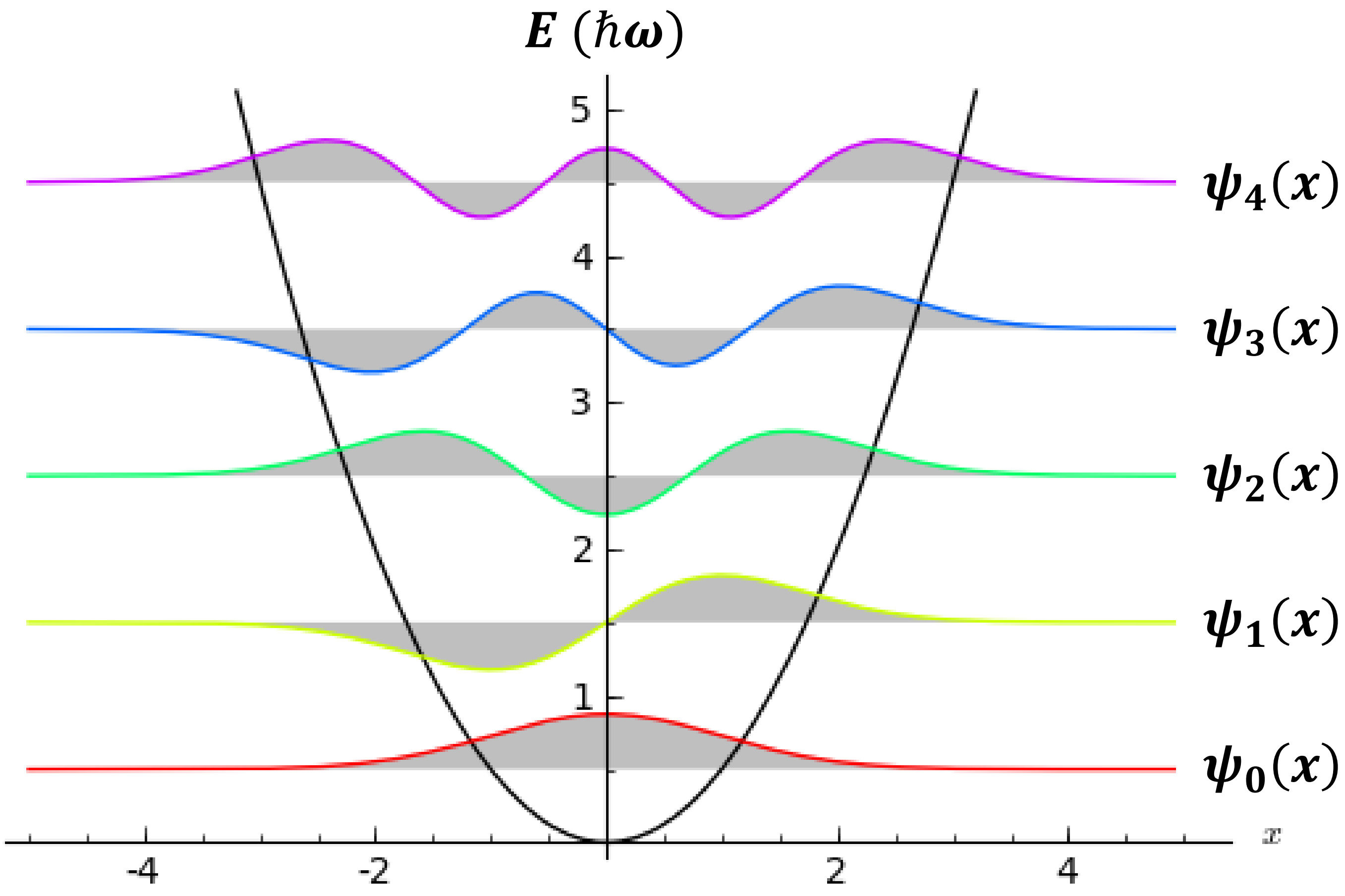
The Hermite polynomials are polynomial functions of x with leading order x^n

A few low-lying wavefunctions:

$$\boldsymbol{\psi_0(x) = A_0\, e^{-m\omega x^2/2\hbar}}$$

$$\boldsymbol{\psi_1(x) = A_1 \sqrt{\frac{m\omega}{\hbar}}\, x\, e^{-m\omega x^2/2\hbar}}$$

$$\boldsymbol{\psi_2(x) = A_2 \left(1 - \frac{2m\omega}{\hbar}\, x^2\right) e^{-m\omega x^2/2\hbar}}$$



Character of the Quantum Harmonic Oscillator Solutions

The potential energy is symmetric: $V(-x) = V(x)$,

This leads to a symmetry in the probability density, $P(-x) = P(x)$

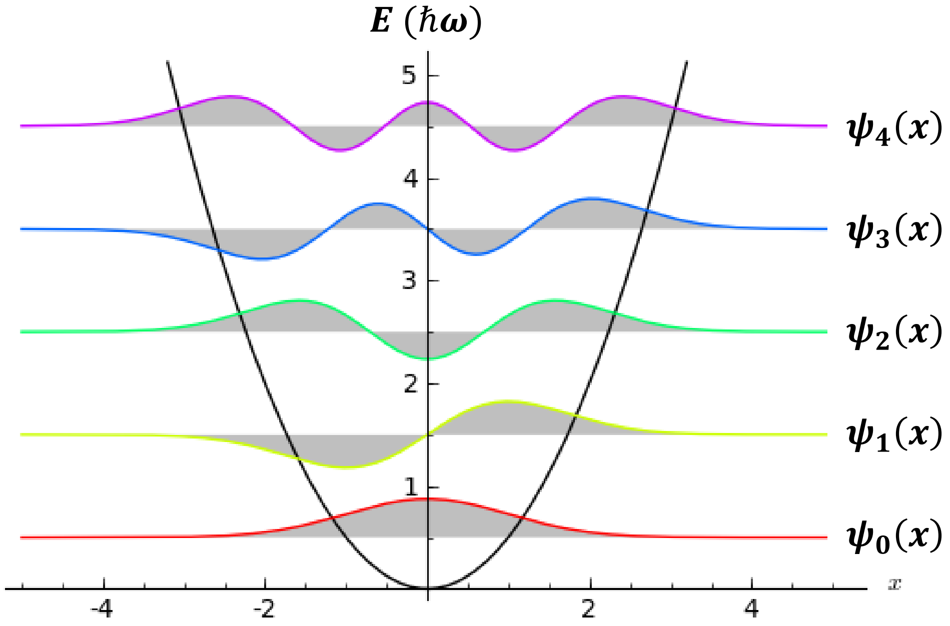
$$|\boldsymbol{\psi}(-\boldsymbol{x})|^2 = |\boldsymbol{\psi}(\boldsymbol{x})|^2$$

$$\boldsymbol{\psi}(-\boldsymbol{x}) = \pm\boldsymbol{\psi}(\boldsymbol{x})$$

Hence the eigenfunctions will have even and odd parity

$\boldsymbol{n}$ **even – symmetric**

$\boldsymbol{n}$ **odd – anti-symmetric**



Character of the Quantum Harmonic Oscillator Solutions

Classically-Allowed vs. Classically-Forbidden Regions

In the classically *allowed* region, $-A < x < A$, the Schrodinger equation can be written as $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}(E - V(x))\psi(x)$

since $E > V(x)$ there the prefactor of $\psi(x)$ on the right hand side is negative

The wavefunction curves *towards* the axis

In the classically *forbidden* region, $x > A, x < -A$, since $E < V(x)$ there the prefactor of $\psi(x)$ on the right hand side is positive

The wavefunction curves *away from* the axis